ground water

Technical Note/

Exact Solution for Two-Dimensional Flow to a Well in an Anisotropic Domain

by Charles R. Fitts[1]

Abstract

Although most current applications of the analytic element method are formulated for isotropic hydraulic conductivity, anisotropic domains can be modeled with analytic elements using the well-known coordinate transformation where one coordinate axis is scaled by the square root of the anisotropy ratio. If the standard analytic solution for steady radial flow to a well is used with this coordinate transformation, the resulting solution correctly models the far field but it does not meet the constant head boundary condition at the well radius. This could be a significant shortcoming if you are interested in the flow field close to the well or want to estimate the head at the pumping well. A new solution for two-dimensional steady flow to a well in an anisotropic domain is presented. This solution satisfies the governing equations exactly and meets the constant head boundary condition at the well radius exactly. It was derived using a conformal mapping.

Introduction

Most current applications of the analytic element method (AEM) assume two-dimensional steady flow in an isotropic domain. Flow in an anisotropic domain may be modeled with these same techniques, except that the analytic functions must be written in terms of scaled coordinates instead of actual coordinates. The necessary coordinate transformation is well-known and involves scaling coordinate axes by the square root of the ratio of greatest transmissivity to least transmissivity. Writing the governing steady-flow equations in terms of scaled coordinates results in the Laplace or Poisson equation. Solutions to these equations can be superposed in the typical AEM fashion.

The Thiem equation, or variations of it, is a solution to the Laplace equation for radial flow to a well (Thiem 1906). When this solution is applied to an anisotropic situation using the standard coordinate transformation, contours of hydraulic head form concentric circles in the transformed coordinates and nested ellipses in the actual coordinates. This is a fine solution, except that it does not meet the constant head condition at the well radius. At the well radius, there should be a circular constant head condition in the actual coordinates and an elliptical constant head condition in the transformed coordinates—just the opposite of what the Thiem solution with transformed coordinates provides.

A new analytic solution is presented that meets the boundary condition at the well perfectly. The far field of this solution is the same as that provided by the Thiem solution with transformed coordinates, but they differ near the wellbore. The precision of this solution near the well is particularly relevant when the head at the well is specified or sought and when modeling flow to large-diameter wells or cylindrical excavations.

Coordinates and Governing Equations

Consider an aquifer that is homogeneous but anisotropic with respect to horizontal hydraulic conductivity and transmissivity. The aquifer can be confined or unconfined, so long as the anisotropy ratio is uniform. For simplicity of presentation, put the origin of a local x, y coordinate system at the center of the well with x and y axes parallel to the minor and major principal axes of transmissivity T_x and T_y, respectively (Figure 1). Under these circumstances, the governing equation for head h in

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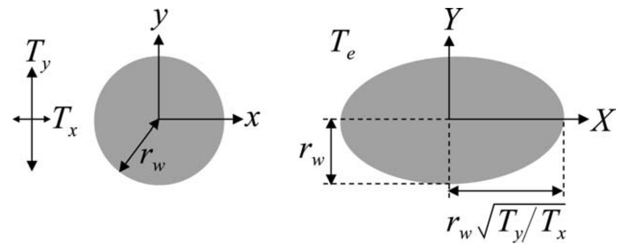

Figure 1. Wellbore, shown shaded, in actual *x*, *y* coordinates (left) and transformed *X*,*Y* coordinates (right).

steady two-dimensional flow with no recharge/leakage is written as follows:

$$\frac{\partial}{\partial x}\left(T_x \frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(T_y \frac{\partial h}{\partial y}\right) = 0 \tag{1}$$

As outlined by Muskat (1937) and Bear and Dagan (1965), define a transformed X, Y coordinate system as follows

$$X = x\sqrt{T_y/T_x},\ Y = y \tag{2}$$

and an equivalent isotropic transmissivity T_e as

$$T_e = \sqrt{T_y T_x} \tag{3}$$

With these definitions, the governing Equation 1 can be rewritten in terms of the transformed X, Y coordinates as

$$\frac{\partial}{\partial X}\left(T_e \frac{\partial h}{\partial X}\right) + \frac{\partial}{\partial Y}\left(T_e \frac{\partial h}{\partial Y}\right) = 0 \tag{4}$$

In x, y coordinates, the well boundary is a circle, but in X, Y coordinates, it is an ellipse (Figure 1).

Using the standard AEM approach, the discharge potential Φ is defined in terms of aquifer parameters and h so that the following equations are true.

$$\frac{\partial \Phi}{\partial X} = T_e \frac{\partial h}{\partial X}, \quad \frac{\partial \Phi}{\partial Y} = T_e \frac{\partial h}{\partial Y} \tag{5}$$

Different $\Phi(h)$ relationships that satisfy Equation 5 have been defined for confined aquifers, unconfined aquifers, coastal aquifers, and stratified aquifers (Strack 1989; Haitjema 1995; Fitts 2002). Combining Equations 4 and 5 results in the Laplace equation for Φ:

$$\frac{\partial^2 \Phi}{\partial X^2} + \frac{\partial^2 \Phi}{\partial Y^2} = 0 \tag{6}$$

Solution

The solution is written as a complex potential $\Omega(Z) = \Phi + i\Psi$ in terms of the complex variable $Z = X + iY$, where Ψ is the stream function. The complex form of the radial flow solution (Thiem 1906) in terms of Z is

$$\Omega(Z) = \frac{Q}{2\pi} \ln Z + D \tag{7}$$

where Q is the discharge of the well and D is a real constant. Although its far field is correct, Equation 7 does not meet the constant Φ boundary condition at the well. With this equation, all equipotentials form circles in the Z plane, so it violates the constant Φ condition at the ellipse-shaped well boundary.

The constant Φ condition can be met perfectly by using a conformal mapping that maps the exterior of the well ellipse in the Z plane onto the exterior of a circle in the U plane (Nehari 1975; Strack 1989). The conformal mapping is given as

$$U = \frac{1}{2}\left(Z + \sqrt{Z-f} \cdot \sqrt{Z+f}\right) \tag{8}$$

where U is a complex variable and f is the focal length of the ellipse that is the well boundary in the Z plane.

$$\begin{aligned} f &= \sqrt{\left(r_w\sqrt{T_y/T_x}\right)^2 - (r_w)^2} \\ &= r_w\sqrt{T_y/T_x - 1} \end{aligned} \tag{9}$$

The two square roots in Equation 8 are not combined because this is how they must be evaluated when the arguments of complex numbers range from $-\pi$ to $+\pi$. At large distances from the well, $|Z| \gg f$ and U approaches Z, as can be seen by examining Equation 8. The exact solution is the radial flow solution in terms of the complex variable U.

$$\Omega(U) = \frac{Q}{2\pi} \ln U + D \tag{10}$$

The solution is evaluated as $\Omega(U(Z(z)))$, where $z = x + iy$, using Equations 2, 8, 9, and 10.

This solution has been programmed in FORTRAN, and example results are presented in Figure 2. It shows the solution in the z and Z planes for $T_y/T_x = 10$. The plot in the Z plane forms a flow net as solutions of the Laplace equation must (the gradients of Φ and Ψ are orthogonal and of the same magnitude). It also shows that the constant head boundary condition at the well is met on a circle in the z plane and on an ellipse in the Z plane. In the z plane, contours of Φ and Ψ intersect at right angles only when they are parallel to either of the principal axes of transmissivity, a characteristic of anisotropic flow nets.

Discussion

The superposition principle that is the core of the AEM works for anisotropic cases, so long as superposition is done in terms of the equivalent isotropic system. The solution for each well is written in terms of transformed local $Z = X + iY$ coordinates with the well at the origin, and then the complex potential is evaluated with Equations 8, 9, and 10. Figure 3 shows a model where

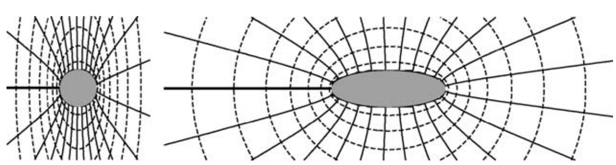

Figure 2. Solution for one well with $T_y/T_x = 10$, showing contours of Φ (dashed) and Ψ (solid). Solution in actual *x*, *y* coordinates (left) and transformed *X*,*Y* coordinates (right).

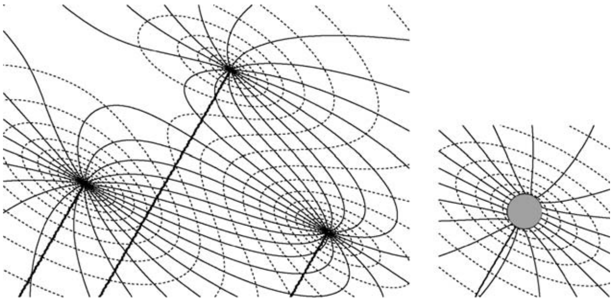

Figure 3. Solution for three wells with $T_y/T_x = 5$, showing contours of Φ (dashed) and Ψ (solid). The middle and left wells extract water ($Q > 0$), and the right well injects water ($Q < 0$). The plot at right shows a close-up of the solution at the lower right well, with the wellbore shaded. The scales of the two plots differ by a ratio of 50:1.

three of these well solutions are superposed in a domain with $T_y/T_x = 5$ and the x axis oriented at an angle of 60° from the horizontal in this plot. The dark lines extending in the negative x direction from each well are branch cuts, necessary discontinuities in the stream function that are associated with discharging elements.

When other solutions are added to a well solution, the constant head condition at the wellbore is no longer perfect; the other solutions cause a slight gradient across the wellbore. This error, typically insignificant, also occurs with well elements in isotropic AEM models.

Although just well elements are shown here, the same concepts can be extended to include line elements and area-sink elements. The same functions that are used in line elements and area-sink elements in isotropic AEM models may be used in the Z plane of anisotropic models. Care must be taken to interpret the discharge rates of these elements correctly since distances and areas are distorted in the Z plane. Adding line and area-sink capabilities will be the next step in this research.

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References

Bear, J., and G. Dagan. 1965. The relationship between solutions of flow problems in isotropic and anisotropic soils. *Journal of Hydrology* 3, no. 2: 88–96.

Fitts, C.R. 2002. *Groundwater Science*. San Diego, California: Academic Press.

Haitjema, H.M. 1995. *Analytic Element Modeling of Groundwater Flow*. San Diego, California: Academic Press.

Muskat, M. 1937. *The Flow of Homogeneous Fluids through Porous Media*. New York: McGraw-Hill.

Nehari, Z. 1975. *Conformal Mapping*. New York: Dover Publications.

Strack, O.D.L. 1989. *Groundwater Mechanics*. Englewood Cliffs, New Jersey: Prentice Hall.

Thiem, G. 1906. *Hydrologische Methoden*. Leipzig, Germany: Gebhardt.