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Three-Dimensional Streamlines in Dupuit-Forchheimer Models

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An approximate method is presented for determining streamlines in three dimensions using two-dimensional Dupuit-Forchheimer models. The latter models of horizontal flow, which may comprise several aquifers, are based on the assumption that the resistance to flow in the vertical direction is neglected. This assumption does not preclude the occurrence of flow in the vertical direction; vertical components of flow are determined in an approximate fashion by requiring continuity of flow. The problem of leakage from a shallow pond above an unconfined aquifer is discussed as an example of determining streamlines in three dimensions. The errors induced by the approximation are determined for a few idealized flow problems that can be solved exactly.

INTRODUCTION

The modeling of steady regional aquifer flow is generally done by the use of two-dimensional numerical or analytical models that are based upon an approximation introduced by *Dupuit* [1863] and *Forchheimer* [1886]. This approximation was made originally for unconfined flow and consists of the assumption that the head is constant over the height of the aquifer. The Dupuit-Forchheimer assumption has been applied to aquifer systems with various degrees of complexity by a score of researchers. *Girinskii* [1946] introduced discharge potentials which make it possible to deal with aquifers where the permeability varies in the vertical direction, and with combined confined and unconfined flow. Girinskii's potentials also were applied to cases of steady interface flow in coastal aquifers, where fresh water flows above salt water at rest [*Girinskii*, 1947; *Strack*, 1976]. *Mjatiev* [1947] (see also *Polubarinova-Kochina* [1962, pp. 385–392]) and *Huisman and Kemperman* [1951] developed a technique for modeling flow in a system of aquifers separated by leaky layers, assuming horizontal flow in the aquifers and vertical flow in the leaky layers. *Strack* [1981] outlined a method for solving problems of flow in a system of two aquifers separated by a thin impermeable layer.

Most of the above applications of the Dupuit-Forchheimer assumption have been implemented in computer models, often generalized to handle transient flow, generally with good results. As the concern for pollution of aquifers increases, it becomes desirable to use the models to predict travel paths of contaminated water through the aquifer system. For steady flow this means that the model must be capable of producing streamlines. The streamline pattern will give a good insight into the spreading of contaminants [see *Nelson*, 1978], provided that the microscopic dispersion may be neglected with respect to the macroscopic one and that the contaminants dissolved in the groundwater do not alter the fluid properties. The models based upon the Dupuit-Forchheimer assumption, however, suffer from the drawback that streamlines can be traced only in the horizontal plane. This deficiency is particularly limiting when dealing with systems of multiple aquifers, where contaminants may travel through interconnections or leaky layers from one aquifer to another. An approximate method for determining streamlines in three-dimensional space, using models based upon the Dupuit-Forchheimer assumption, is presented in this paper. The basic idea of this method is to interpret the Dupuit-Forchheimer assumption as the assumption that the resistance to flow, rather than the flow itself, in the vertical direction is neglected. Thus flow may occur in the vertical direction but is not associated with any variation of the head. The approximation will be adequate only if the aquifer is thin with respect to its lateral extent. To avoid the potential confusion arising from using a two-dimensional Dupuit-Forchheimer formulation for suitable three-dimensional flows, such flows will be referred to in this paper as shallow two-dimensional flows. Both streamlines and fronts of constant travel time obtained in this way are compared with streamlines and isochrones determined from exact solutions for a few idealized flow problems.

Flow in the vertical direction in Dupuit-Forchheimer models was considered earlier by *Streltsova* [1972] and by *De Josselin de Jong* [1981]. Streltsova used a "linear equation of vertical transfer" to estimate the difference between the average head to the value at the free surface for unsteady radial flow in an unconfined aquifer. De Josselin de Jong used approximated vertical components of flow in order to determine the motion of an interface between fresh and salt water.

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SHALLOW FLOW WITH A CONSTANT RATE OF INFILTRATION

The method will be introduced for a case of one-dimensional shallow flow in a semi-infinite aquifer of constant thickness H, where infiltration occurs at a constant rate N $[L/T]$ along the entire upper boundary of the aquifer. The base of the aquifer is horizontal and impermeable, and the left boundary is vertical and impermeable (see Figure 1). There is no flow normal to the plane of the drawing. Cartesian x, z coordinates are introduced with the x axis coinciding with the base and the z axis pointing vertically upward along the left boundary. Use is made of the discharge potential Φ, defined as

$$\Phi = kH\phi \tag{1}$$

where k is the permeability $[L/T]$. The differential equation for the potential is

$$d^2\Phi/dx^2 = -N \tag{2}$$

so that

$$\Phi = -\tfrac{1}{2}Nx^2 + Ax + B \tag{3}$$

where A and B are constants of integration. The discharge Q_x represents the amount of flow passing through a cross section, i.e.,

$$Q_x = q_x H \tag{4}$$

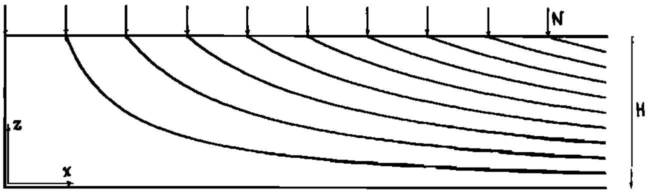

Fig. 1. Shallow confined flow with constant infiltration.

where q_x is the specific discharge vector $[L/T]$. Since

$$Q_x = -d\Phi/dx \tag{5}$$

it follows that

$$Q_x = Nx + A \tag{6}$$

and the condition that $Q_x = 0$ at $x = 0$ implies that $A = 0$. Hence (3) becomes

$$\Phi = -\tfrac{1}{2}Nx^2 + \Phi_0 \tag{7}$$

where Φ_0 is the value of Φ at $x = 0$.

The Dupuit-Forchheimer assumption implies that the head ϕ is constant over the height of the aquifer. Hence the specific discharge q_x is constant over the height of the aquifer, an assumption that was tacitly made in writing (4). It is this assumption that makes it possible to determine the elevation of a fluid particle that entered the aquifer at a point P (x_0, H) (see Figure 2). The discharge $Q_x^{(t)}$ that flows above the streamline through P can be computed at any point x as the amount of infiltration that occurred over the distance $x - x_0$:

$$Q_x^{(t)} = N(x - x_0) \tag{8}$$

The total discharge Q_x at x is obtained from (6), with $A = 0$, as

$$Q_x = Nx \tag{9}$$

and the discharge below the streamline at x is $Q_x^{(b)}$ with

$$Q_x^{(b)} = Q_x - Q_x^{(t)} = Nx - Nx + Nx_0 = Nx_0 \tag{10}$$

Because the fluid is assumed to be uniformly distributed over the height of the aquifer at x, the elevation Z of the streamline above the base is found from

$$Z/H = Q_x^{(b)}/Q_x = x_0/x \tag{11}$$

As is discussed further below, (11) is identical to the expression obtained from the exact solution to the problem of Figure 1. It may be noted, however, that this exact correspondence only occurs when both the infiltration rate and the aquifer thickness are constant.

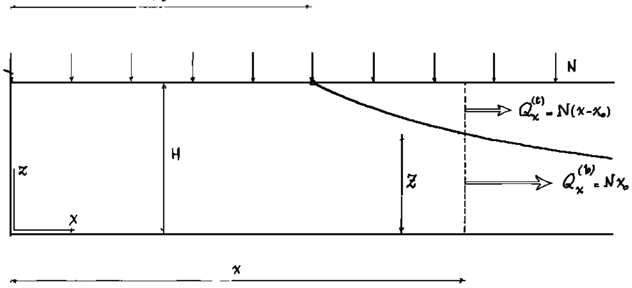

Fig. 2. The path of a fluid particle.

SHALLOW UNCONFINED FLOW WITH CONSTANT INFILTRATION

The above technique is applicable equally well to the case of an unconfined aquifer. The expression for the discharge potential for that case is

$$\Phi = \tfrac{1}{2}k\phi^2 \tag{12}$$

but the expression of Φ in terms of x and y is unchanged and equals (7). The expression for the streamline through x_0 becomes

$$Z/h(x) = x_0/x \tag{13}$$

where h is the saturated thickness of the aquifer. It is noted that h is a function of x for this case.

THE EQUATION FOR A STREAMLINE

The analysis given above is applicable only to shallow one-dimensional flow. An expression for the elevation Z of a streamline will be derived below for shallow two-dimensional flow. This will be done for the general case that there is an infiltration rate $N^{(t)}$ along the upper boundary of an aquifer of variable saturated thickness h and an infiltration rate $N^{(b)}$ along the horizontal lower boundary (see Figure 3). Both infiltration rates may vary with position. Cartesian coordinates x_1, x_2 are introduced in the horizontal plane. It is assumed that the solution to the problem of shallow flow is known, so that the components $Q_i = (Q_1, Q_2)$ of the discharge vector are known in terms of x_i $(i = 1, 2)$. The discharge vector, then, will satisfy the equation of continuity

$$\partial Q_i/\partial x_i = N^{(t)} + N^{(b)} \tag{14}$$

where the Einstein summation convention is adopted, i.e., summation is implied over any index that occurs twice in a term. The discharge $Q_i^{(t)}$, flowing above a streamline, must fulfill the equation of continuity

$$\partial Q_i^{(t)}/\partial x_i = N^{(t)} \tag{15}$$

Similarly, the discharge $Q_i^{(b)}$, flowing below a streamline, satisfies

$$\partial Q_i^{(b)}/\partial x_i = N^{(b)} \tag{16}$$

The three discharge vectors are parallel to one another at each point of the aquifer, so that

$$Q_i^{(b)} = \mu Q_i \qquad Q_i^{(t)} = (1 - \mu)Q_i \tag{17}$$

where μ is a scalar function of x_i. Substitution of the expression in (17) for $Q_i^{(b)}$ in (16) yields

$$\frac{\partial(\mu Q_i)}{\partial x_i} = Q_i \frac{\partial \mu}{\partial x_i} + \mu \frac{\partial Q_i}{\partial x_i} = N^{(b)} \tag{18}$$

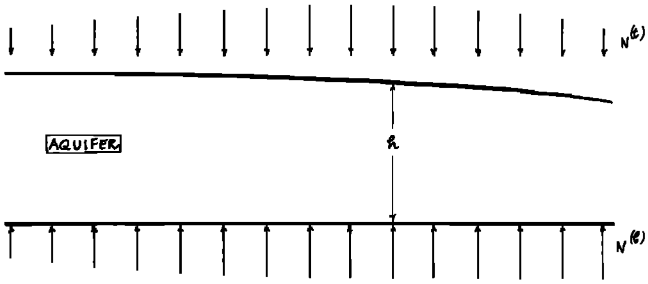

Fig. 3. Shallow flow in an aquifer with inflow through the upper and lower boundaries.

Using (14), this becomes

$$Q_i \frac{\partial \mu}{\partial x_i} + \mu[N^{(b)} + N^{(t)}] = N^{(b)} \tag{19}$$

Let s be the arc length traced along the projection of a streamline on a horizontal plane. Then the unit vector

$$e_i = dx_i/ds \tag{20}$$

points in the direction of Q_i at point x_i of the projected streamline defined by

$$x_i = x_i(s) \tag{21}$$

If the component of Q_i along the streamline is denoted as Q_s, then

$$Q_i = Q_s\, dx_i/ds \tag{22}$$

and using this expression in (19), the following is obtained:

$$Q_s \frac{d\mu}{dx_i}\frac{dx_i}{ds} + \mu[N^{(b)} + N^{(t)}] = N^{(b)} \tag{23}$$

The first term equals $Q_s\, d\mu/ds$, so that (23) becomes, writing $N^{(b)} + N^{(t)}$ as N,

$$Q_s(d\mu/ds) + \mu N = N^{(b)} \tag{24}$$

where

$$N = N^{(b)} + N^{(t)} \tag{25}$$

This differential equation may be integrated, and the solution is [see *Spiegel*, 1968]

$$\mu(s) = e^{-f(s)}\left[\mu(s_0) + \int_{s_0}^{s} \frac{N^{(b)}}{Q_s} e^{f(s)}\, ds\right] \tag{26}$$

where s_0 is the starting point of the streamline, $\mu(s_0)$ is the value of μ at s_0, and the function $f(s)$ is defined as

$$f(s) = \int_{s_0}^{s} \frac{N}{Q_s}\, ds \tag{27}$$

The expression for Z, the elevation of the streamline above the base, is obtained by noticing that uniformity of the specific discharge vector over the saturated thickness implies that

$$Z/h = Q_s^{(b)}/Q_s = \mu \tag{28}$$

Writing $\mu(s_0)$ as $Z(s_0)/h(s_0)$, (26) becomes

$$\frac{Z(s)}{h(s)} = e^{-f(s)}\left[\frac{Z(s_0)}{h(s_0)} + \int_{s_0}^{s} \frac{N^{(b)}}{Q_s} e^{f(s)}\, ds\right] \tag{29}$$

As an example, the expression for the streamline (13) will be obtained from (29). For that case, $N^{(b)} = 0$, and $s = x$, so that

$$f(s) = \int_{x_0}^{x} \frac{N}{Nx}\, dx = \ln \frac{x}{x_0} \tag{30}$$

where use is made of (9). Substitution in (29) yields

$$Z/h = (Z_0/h_0)/(x_0/x) \tag{31}$$

If the streamline starts at the phreatic surface, so that $Z_0 = h_0$, (31) reduces to (13).

APPROXIMATE EXPRESSION FOR q_z

An expression for the vertical component q_z of the specific discharge vector may be obtained from (29) as follows. The derivative of Z with respect to s yields the inclination of the streamline to the horizontal plane. Differentiation of (29) gives

$$\frac{dZ}{ds} = \frac{Z}{h}\frac{dh}{ds} + h\left\{-\frac{N}{Q_s} e^{-f(s)}\left[\frac{Z(s_0)}{h(s_0)} + \int_{s_0}^{s} \frac{N^{(b)}}{Q_s} e^{f(s)}\, ds\right] + \frac{N^{(b)}}{Q_s}\right\} \tag{32}$$

The first term between the braces equals $-(N/Q_s)(Z/h)$, so that

$$\frac{dZ}{ds} = \frac{1}{h}\frac{dh}{ds} Z - \frac{N}{Q_s} Z + h\frac{N^{(b)}}{Q_s} \tag{33}$$

Furthermore,

$$dZ/ds = q_z/q_s = hq_z/Q_s \tag{34}$$

so that, with (33),

$$q_z = \frac{Q_s}{h^2}\frac{dh}{ds} Z - \frac{N}{h} Z + N^{(b)} \tag{35}$$

or

$$q_z = -Q_s \frac{dh^{-1}}{ds} Z - \frac{N}{h} Z + N^{(b)} \tag{36}$$

It appears, as one might expect, that for an aquifer of constant thickness $h = H =$ const, q_z varies linearly between the values $-N + N^{(b)} = -N^{(t)}$ along the upper boundary and $N^{(b)}$ along the lower boundary of the aquifer. It is to be noted that the above expression for q_z is not associated with any variation of the head ϕ in the vertical direction; the solution has been determined by neglecting resistance to flow in the vertical direction. The component q_z is generated solely by considerations of continuity of flow, rather than by solving a three-dimensional boundary value problem. Exact expressions for q_z will therefore be obtained only for a few simple cases. Such cases occur when the three-dimensional solution yields specific discharge components q_x and q_y that are constant over the height of an aquifer, an assumption mentioned repeatedly in the above analysis. The expression (36) for q_z may, however, be useful for determining path lines for cases of transient flow, where all three components of velocity are integrated with respect to time.

LEAKAGE FROM A CIRCULAR POND

The problem of infiltration through the bottom of a circular pond of radius R above an unconfined aquifer is chosen as an application of the technique described above and is illustrated in Figure 4. Contaminated water leaks at a constant rate through the bottom of the pond, which lies closely above the water table. It is assumed that partially saturated flow can be ignored. The contaminated water has the same physical properties as the groundwater. The problem is to estimate the shape and position of the stream surfaces that bound the plume of contaminated groundwater in space. It will be assumed at first that the flow of contaminated water has occurred over an infinitely long period. It may be noted that this problem resembles that studied by *Nelson* [1978], who considered the case that the pond is fully penetrating so that the plume extended over the full aquifer thickness.

The flow is unconfined; the discharge potential Φ for unconfined flow is

$$\Phi = \tfrac{1}{2}k\phi^2 \tag{37}$$

and the components Q_x and Q_y of the discharge vector are obtained as

$$Q_x = -\partial\Phi/\partial x \qquad Q_y = -\partial\Phi/\partial y \tag{38}$$

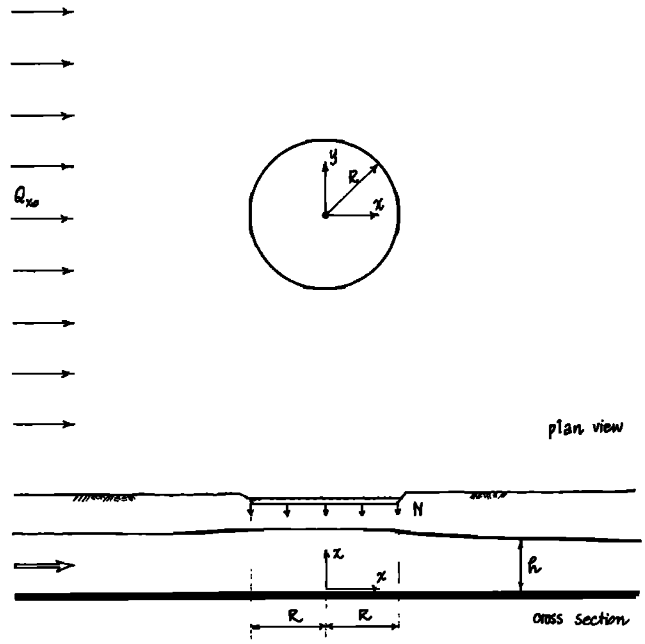

Fig. 4. Leakage from a circular pond.

The origin of the Cartesian x, y coordinate system in the horizontal plane is chosen at the center of the pond. The differential equation for the potential Φ for unconfined flow with infiltration at a rate N $[L/T]$ is the Poisson equation

$$\nabla^2\Phi = -N \qquad x^2 + y^2 \leq R^2 \tag{39}$$

and applies in the aquifer below the pond. Outside the pond the potential is harmonic:

$$\nabla^2\Phi = 0 \qquad x^2 + y^2 > R^2 \tag{40}$$

Without the pond present, a uniform flow occurs with a discharge Q_{x0} in the x direction. The potential for the flow problem defined above may be written in the following form:

$$\Phi = -Q_{x0}x + NF(x, y) + \Phi_0 \tag{41}$$

where Φ_0 is the potential at the origin and $F(x, y)$ is defined as

$$\begin{aligned} F(x, y) &= -\tfrac{1}{4}(x^2 + y^2) \qquad x^2 + y^2 \leq R^2 \\ F(x, y) &= -\frac{R^2}{4}\left[\ln\frac{x^2 + y^2}{R^2} + 1\right] \qquad x^2 + y^2 > R^2 \end{aligned} \tag{42}$$

This function has the following properties. Underneath the pond, $\nabla^2 F = -1$, so that $N\nabla^2 F = -N$, whereas outside the pond, $\nabla^2 F = 0$. Hence (39) and (40) are met by (41). Furthermore, F and dF/dr, where r is the radial coordinate, are continuous across the boundary of the pond, so that both the potential Φ and the component of the discharge vector normal to the boundary of the pond are continuous. Equation (41) therefore is the solution to the problem.

The Stagnation Points

The locations of the stagnation points are obtained by setting the components Q_x and Q_y equal to zero. This gives, with (38), (41), and (42), the following expressions for the coordinates x_s and y_s of the stagnation points:

$$Q_{x0} + \tfrac{1}{2}Nx_s = 0 \qquad \tfrac{1}{2}Ny_s = 0 \qquad x_s^2 + y_s^2 \leq R^2 \tag{43}$$

and

$$\begin{aligned} Q_{x0} + \frac{NR^2}{2}\frac{x_s}{x_s^2 + y_s^2} = 0 \qquad \frac{NR^2}{2}\frac{y_s}{x_s^2 + y_s^2} = 0 \\ x_s^2 + y_s^2 \geq R^2 \end{aligned} \tag{44}$$

It follows that stagnation points may lie only on the x axis,

$$y_s = 0 \tag{45}$$

and (43) and (44) become

$$2Q_{x0}/(NR) + (x_s/R) = 0 \qquad |x_s| \leq R \tag{46}$$

$$2Q_{x0}/(NR) + (R/x_s) = 0 \qquad |x_s| \geq R \tag{47}$$

Neither one of these equations yields a solution when

$$2Q_{x0}/(NR) > 1 \tag{48}$$

that is, when no stagnation points exist, because (46) would yield an $|x_s|$ greater than R and (47) an $|x_s|$ less than R. If

$$2Q_{x0}/(NR) \leq 1 \tag{49}$$

stagnation points exist, and stagnation does occur at the points

$$x_{s_1} = -2Q_{x0}/N \qquad x_{s_1} \geq -R \tag{50}$$

$$x_{s_2} = -NR^2/(2Q_{x0}) \qquad x_{s_2} \leq -R \tag{51}$$

These two points coincide when the equals sign in (49) applies.

Vertical Stream Surfaces

It is a consequence of the Dupuit-Forchheimer assumption that each streamline lies on a single vertical surface. The intersections of these vertical stream surfaces with the aquifer base are curves which may be considered as streamlines for the discharge vector (Q_x, Q_y) rather than for the specific discharge vector (q_x, q_y, q_z). They will be referred to below as discharge streamlines.

Outside the pond the potential (41) is harmonic, and the equations for the discharge streamlines may be obtained by letting the stream function Ψ be constant. The latter function is the conjugate harmonic of Φ and has the following form:

$$\Psi = -Q_{x0}y - \frac{NR^2}{2}\theta \qquad x^2 + y^2 > R^2 \tag{52}$$

where

$$\theta = \arctan(y/x) \qquad -\pi < \theta \leq \pi \tag{53}$$

The stream function does not exist inside the pond, and the equations for the discharge streamlines are obtained by integration of the following equation:

$$Q_x\,dy - Q_y\,dx = 0 \tag{54}$$

where Q_x and Q_y are obtained upon differentiation of (41),

$$Q_x = Q_{x0} + \frac{N}{2}x \qquad Q_y = \frac{N}{2}y \tag{55}$$

Integration of (54), with (55), yields the following equation for a straight line:

$$y = [x - x_s^{(i)}]\tan\beta \qquad x^2 + y^2 \leq R^2 \tag{56}$$

where β is the angle between the discharge streamline and the x axis and $x_s^{(i)}$ is defined as

$$x_s^{(i)} = -2Q_{x0}/N \tag{57}$$

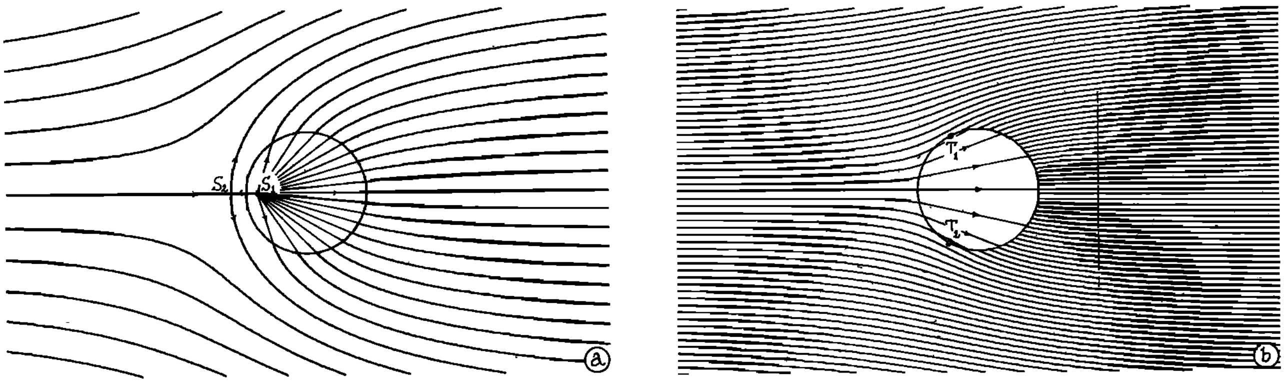

Fig. 5. Discharge streamlines for cases (*a*) with and (*b*) without stagnation.

If (49) applies, stagnation occurs below the pond at $(x_s^{(i)}, 0)$ (see (50)); otherwise $(x_s^{(i)}, 0)$ represents a point upstream from the pond. The streamlines inside the pond are straight lines through $x_s^{(i)}$ and outside the pond are given by (52) with Ψ constant.

Discharge streamlines are reproduced in Figure 5 for the cases that stagnation occurs (Figure 5*a*) and no stagnation points exist (Figure 5*b*). The discharge streamlines that bound the plume, the dividing discharge streamlines, may be identified in Figure 5 as follows. For the case in Figure 5*a* these dividing curves pass through the stagnation point S_2 outside the pond; for the case in Figure 5*b* they are the curves that touch the pond at the points T_1 and T_2. The data for the case of Figure 5*a* are $N/k = 0.001$, $Q_{x0}/(NR) = 0.4$, and $\phi_0/R = 0.2$,; the data for the case of Figure 5*b* are $N/k = 0.001$, $Q_{x0}/(NR) = 1$, and $\phi_0/R = 0.2$.

Streamlines

The equations for the streamlines in each vertical surface through the discharge streamlines are obtained by the use of (29). Let s be the length measured along a discharge streamline below the pond, and let s be zero at $x = x_s^{(i)}$, $y = 0$. Then one obtains, with (56),

$$s = \frac{x - x_s^{(i)}}{\cos\beta} = \frac{y}{\sin\beta} \tag{58}$$

and the expressions (55) for Q_x and Q_y may be written in terms of s as follows:

$$Q_x = \frac{N}{2}x + Q_{x0} = \frac{N}{2}[x - x_s^{(i)}] = \frac{N}{2}s\cos\beta$$
$$Q_y = \frac{N}{2}y = \frac{N}{2}s\sin\beta \qquad x^2 + y^2 \leqq R^2 \tag{59}$$

The component of the discharge vector along s becomes

$$Q_s = (Q_x^2 + Q_y^2)^{1/2} = \frac{N}{2}s \qquad x^2 + y^2 \leqq R^2 \tag{60}$$

The elevation Z of a point with coordinates x and y on a discharge streamline starting at $s = s_0$ is obtained by the use of (29), with $N^{(t)} = N$ and $N^{(b)} = 0$:

$$\frac{Z(s)}{h(s)} = \frac{Z(s_0)}{h(s_0)}e^{-f(s)} \qquad x^2 + y^2 \leqq R^2 \tag{61}$$

where

$$f(s) = \int_{s_0}^{s}\frac{N}{Q_s}\,ds = \int_{s_0}^{s}\frac{2\,ds}{s} = \ln\left[\frac{s}{s_0}\right]^2 \qquad x^2 + y^2 \leqq R^2 \tag{62}$$

Using this expression for $f(s)$, (61) becomes

$$\frac{Z(s)}{h(s)} = \frac{Z(s_0)}{h(s_0)}\left[\frac{s_0}{s}\right]^2 \qquad x^2 + y^2 \leqq R^2 \tag{63}$$

For the case of Figure 5*a*, all discharge streamlines contained between the dividing discharge streamlines emanate from point S_1 $(x_s^{(i)}, 0)$ inside the pond. Therefore the vertical surfaces through these discharge streamlines each contain one streamline that starts at S_1. It follows from (58) that s is zero at S_1. Furthermore, all streamlines starting at $s = s_0 = 0$ lie on the aquifer base, since Z/h in (63) vanishes for $s_0 = 0$. Hence for the case of Figure 5*a* the contaminated water fills the entire space contained between the dividing discharge streamlines, the phreatic surface, and the aquifer base. The plot of streamlines in a vertical plane through the x axis reproduced in Figure 6*a* corresponds to the case of Figure 5*a*. The plot extends from $x/R = -1.25$ to $x/R = 1.25$ and contains the two stagnation points, which lie at $x/R = -1.25$ and $x/R = -0.8$. The streamlines start at equal intervals between $x/R = -1$ and $x/R = 1$; there are 21 streamlines. A similar plot is shown in Figure 6*b* and corresponds to the case of Figure 5*b*. For this case, stagnation does not occur; $x_s^{(i)}/R = -2$, and the bottom of the plume is marked by the lowest curved streamline. The plot extends between $x/R = -1$ and $x/R = 1.5$. It may be noted that a discontinuity in slope of the streamlines occurs below the boundary of the pond. This discontinuity results from coupling the component q_z directly to the infiltration rate, which is discontinuous across the boundary of the pond.

Only when a stagnation point exists outside the pond will the plume extend over the full height of the aquifer; otherwise the bottom of the plume is a curved surface above the base. A cross section through the plume parallel to the y, z plane is shown in Figure 7. The location of this cross section is marked by the dotted line in Figure 5*b* and is at $x/R = x_1/R = 2$. Points of the curved bottom of the plume are determined as follows. First, the width of the plume at the given location is determined by intersecting the dividing discharge streamlines with the plane $x/R = x_1/R$. Second, each discharge streamline

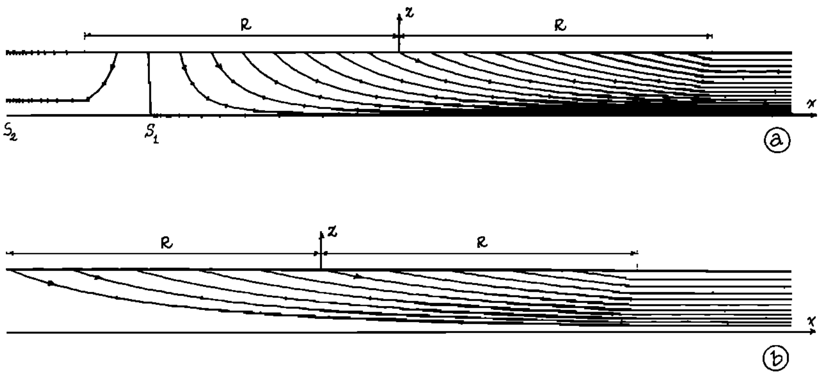

Fig. 6. Streamlines in the x, z plane.

$\Psi = \Psi_j$ inside the plume is intersected with the downstream side of the circle, which yields a value for $s = s_j$, with the aid of (58). Third, the discharge streamline is traced below the pond until it intersects the circle on the upstream side, giving s_{0j}. Fourth, s_j and s_{0j} are substituted for s and s_0 in (63), which yields the elevation Z_j/h_j of the bottom of the plume at s_j. The elevation of the plume at the desired cross section is found, finally, as a function of the saturated thickness h,

$$Z/h = Z_j/h_j \tag{64}$$

which applies along the discharge streamline outside the pond where $N = 0$ (see (29)). The above procedure is implemented in an elementary computer program, written merely for the purpose of illustration. This program is used to show, in Figure 8, how the cross section of the plume varies as a function of $x_s^{(i)}/R$. When $x_s^{(i)}/R = -1$, the plume is rectangular and fills the entire space between the dividing discharge streamlines. As the ratio $2Q_{x0}/(NR) = -x_s^{(i)}/R$ increases, the area of the plume decreases; the plume becomes both shallower and narrower. The data used for this case are the same as those for Figure 5b, except that the saturated aquifer thickness was increased ($\phi_0/H = 1$) in order to make the curves more visible.

The plume attains the geometry described above after the pollution has occurred for a long time. It is possible to determine the front of the plume as it moves through the aquifer as a function of the time recorded since the pollution began. The marks shown on the streamlines in Figure 6 correspond to constant intervals of the dimensionless time $t^* = Nt/(nR)$, where n is the porosity. The front of the plume at any time, therefore, is obtained by connecting corresponding marks.

Implementation of the technique in Dupuit-Forchheimer models amounts to evaluation of (29) while tracing discharge streamlines. It should be noted that the technique is approximate and gives a simplified picture of the real streamline pattern. For example, the vertical faces of the plume shown in Figure 8 for $x_s^{(i)}/R = -1$ do not occur in reality, because the two horizontal components of the specific discharge vector vary greatly over the thickness of the aquifer near the bottom of the pond. For cases where three-dimensional effects are too important to be neglected, a truly three-dimensional solution may be imbedded inside the Dupuit-Forchheimer model.

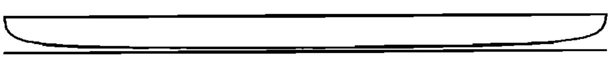

Fig. 7. Cross section through the plume at $x/R = 2$.

VERIFICATIONS

The errors caused by the simplification in the technique presented above will be estimated by comparison with a few exact solutions below. It may be noted that the problems chosen are fictitious and should not be interpreted in terms of flow in real aquifers. The solutions are exact; the resistance to flow in the vertical direction is taken into account.

Axisymmetric Flow With Uniform Infiltration

The first problem is that of axisymmetric flow in an infinite aquifer of constant thickness H and with a horizontal impervious base. Infiltration occurs at a rate N along the entire upper boundary. The origin of an r, z system of cylindrical coordinates is chosen with r on the aquifer base, as shown in Figure 9. The exact solution may be expressed in terms of the head ϕ as the following harmonic function:

$$\phi = -N/(4kH)\,(r^2 - 2z^2) + C \tag{65}$$

where C is a constant. The components q_r and q_z of the specific discharge vector in the r and z directions are obtained from Darcy's law:

$$q_r = +N/(2H)r \qquad q_z = -(N/H)z \tag{66}$$

It appears that the boundary conditions along $z = H$ ($q_z = -N$) and along $z = 0$ ($q_z = 0$) are met and that $q_r = 0$ at $r = 0$. The equations for the streamlines in the r, z plane are obtained from

$$dz/dr = q_z/q_r = -2z/r \tag{67}$$

so that

$$z/z_0 = [r_0/r]^2 \tag{68}$$

where (r_0, z_0) denotes the starting point of the streamline.

The approximate solution has the form

$$\Phi = -\tfrac{1}{4}Nr^2 + C \tag{69}$$

where C is a constant and $\Phi = kH\phi$. Application of (29) with $h(s) = H$, $s = r$, and $Q_r = \tfrac{1}{2}Nr$ yields (68); the approximate expression for the streamlines is identical to the exact one for this case of uniform infiltration. The reader may verify in a similar fashion that the same holds true for the case illustrated in Figure 1.

Complex Variable Approach

The remaining problems discussed in this paper are all two dimensional. The exact solutions will be determined by the use

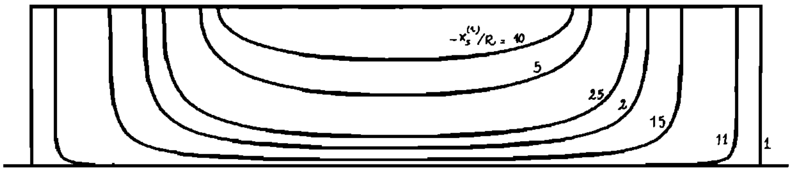

Fig. 8. Cross sections through the plume for various values of $x_s^{(i)}/R = -2Q_{x_0}/NR$.

of complex variables. The complex variable $z = x + iy$ is introduced in the physical plane, where y is the vertical coordinate and where z is not to be confused with the coordinate z used for three-dimensional flow earlier in this paper. The complex potential Ω is introduced as

$$\Omega = k\phi + i\Psi \tag{70}$$

where ϕ is the head and Ψ is the stream function. This complex potential is an analytic function of z, and the specific discharge function w,

$$w = q_x - iq_y = -d\Omega/dz \tag{71}$$

may be used to determine the components q_x and q_y of the specific discharge vector.

Exact Solution for Two-Dimensional Flow in a Layered Aquifer

Consider the problem of flow in an aquifer consisting of two layers of different permeabilities shown in Figure 10. The aquifer is a semi-infinite strip, and the x and y axes form the left-hand and bottom boundaries; both are impermeable. Uniform infiltration occurs at a rate N along the upper boundary, which is parallel to the base. The boundary conditions may be expressed in terms of the components q_x and q_y of the specific discharge vector as

$$\begin{array}{lll} 0 \leq x \leq \infty & y = 0 & q_y = 0 \\ x = 0 & 0 \leq y \leq H^{(1)} + H^{(2)} & q_x = 0 \\ 0 \leq x < \infty & y = H^{(1)} + H^{(2)} & q_y = -N \end{array} \tag{72}$$

where $H^{(1)}$ and $H^{(2)}$ are the thicknesses of the upper and lower layers, respectively. The permeabilities are denoted as $k^{(1)}$ and $k^{(2)}$, and the transmissivity T is defined as

$$T = k^{(1)}H^{(1)} + k^{(2)}H^{(2)} \tag{73}$$

The complex potentials for layers (1) and (2) are given below without derivation:

$$\Omega^{(1)} = -\frac{1}{2}\frac{k^{(1)}}{T}Nz^2 - i\frac{k^{(2)}H^{(2)}}{T}N\left[1 - \frac{k^{(1)}}{k^{(2)}}\right] \cdot [z - iH^{(2)}] + \frac{k^{(1)}}{k^{(2)}}C \tag{74}$$

and

$$\Omega^{(2)} = -\frac{1}{2}\frac{k^{(2)}}{T}Nz^2 + C \tag{75}$$

where C is an arbitrary real constant. The above complex potentials must fulfill the condition along $y = H^{(2)}$ that the head is single valued,

$$0 \leq x \leq \infty \qquad y = H^{(2)} \qquad \Phi^{(2)} = \frac{k^{(2)}}{k^{(1)}}\Phi^{(1)} \tag{76}$$

where use is made of (70): $\Phi^{(2)} = k^{(2)}\phi$ and $\Phi^{(1)} = k^{(1)}\phi$. The reader may verify that (74) and (75) fulfill (76). The boundary conditions (72) are also satisfied, as may be seen by the use of the specific discharge functions $w^{(1)}$and $w^{(2)}$,

$$w^{(1)} = q_x^{(1)} - iq_y^{(1)} = \frac{k^{(1)}}{T}Nz + i\frac{k^{(2)}H^{(2)}}{T}N - i\frac{k^{(1)}H^{(2)}}{T}N \tag{77}$$

and

$$w^{(2)} = q_x^{(2)} - iq_y^{(2)} = \frac{k^{(2)}}{T}Nz \tag{78}$$

Finally, the vertical component of the specific discharge function must be continuous at $y = H^{(2)}$; that this condition is met may be seen by substituting $H^{(2)}$ for y in (77) and (78).

Equations for the streamlines in the two layers are obtained as follows. The value of $\Psi^{(1)}$ along the streamline through $z = z_0 = x_0 + i(H^{(1)} + H^{(2)})$ is $-Nx_0$, as is seen from (74). The equation for the streamline in layer (1) through z_0 is thus obtained by setting the imaginary part of $\Omega^{(1)}$ equal to $-Nx_0$, which yields after some rearrangement

$$\frac{y - H^{(2)}}{H^{(1)}} = -\frac{k^{(2)}H^{(2)}}{k^{(1)}H^{(1)}} + \frac{T}{k^{(1)}H^{(1)}}\frac{x_0}{x} \qquad H^{(2)} \leq y \leq H^{(1)} + H^{(2)} \tag{79}$$

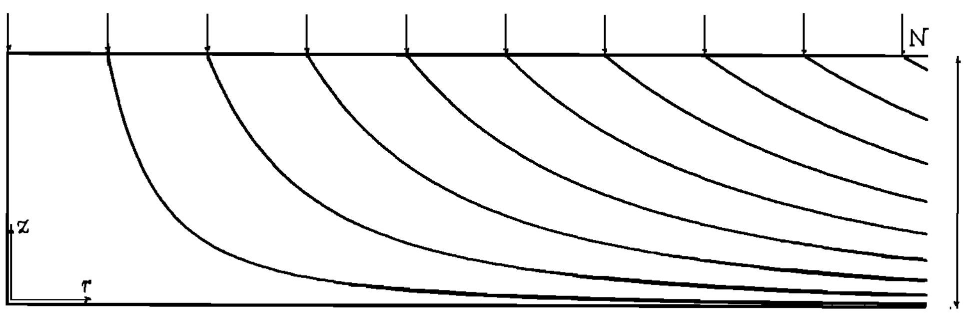

Fig. 9. Axisymmetric flow with uniform infiltration.

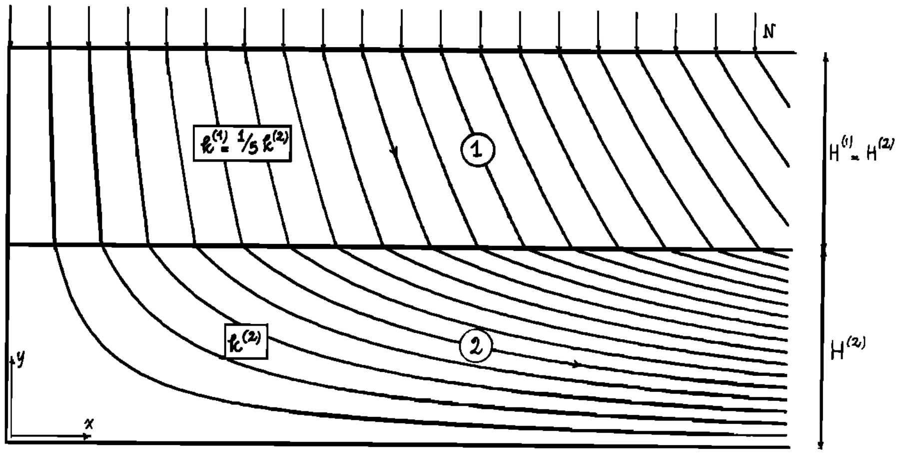

Fig. 10. Flow in a layered aquifer.

The expression for the streamline in layer (2) is obtained in a similar fashion:

$$\frac{y}{H^{(2)}} = \frac{T}{k^{(2)}H^{(2)}}\frac{x_0}{x} \qquad 0 \leqq y \leqq H^{(2)} \tag{80}$$

The two streamlines have in common the point $z = x_0T/(k^{(2)}H^{(2)}) + iH^{(2)}$ of the boundary between the layers. The streamlines are shown in Figure 10 for the case that $H^{(1)} = H^{(2)}$ and $k^{(2)} = 5k^{(1)}$.

Approximate Solution for Two-Dimensional Flow in a Layered Aquifer

Besides being a verification the present case of flow in a layered aquifer serves as an example of determining streamlines in aquifers that contain layers of slightly different permeabilities. The discharge potentials introduced by *Girinskii* [1947] (see also *Aravin and Numerov* [1965]) are applicable to such aquifers, provided that the differences in permeability are not too large. The head, then, may be taken constant over the height of the aquifer. The discharge potentials Φ, $\Phi^{(1)}$, and $\Phi^{(2)}$ are introduced for layers the entire aquifer and (1) and (2):

$$\Phi = T\phi \qquad \Phi^{(1)} = k^{(1)}H^{(1)}\phi \qquad \Phi^{(2)} = k^{(2)}H^{(2)}\phi \tag{81}$$

The corresponding discharge vectors are

$$Q_x = Q_x^{(1)} + Q_x^{(2)} = -\frac{d\Phi}{dx}$$
$$Q_x^{(1)} = -\frac{d\Phi^{(1)}}{dx} \qquad Q_x^{(2)} = -\frac{d\Phi^{(2)}}{dx} \tag{82}$$

It follows from (81) and (82) that

$$Q_x^{(1)} = \lambda^{(1)}Q_x \qquad Q_x^{(2)} = (1-\lambda^{(1)})Q_x = \lambda^{(2)}Q_x \tag{83}$$

where

$$\lambda^{(1)} = \frac{k^{(1)}H^{(1)}}{T} \qquad \lambda^{(2)} = \frac{k^{(2)}H^{(2)}}{T} \tag{84}$$

Furthermore, let $N^{(1)}$ and $N^{(2)}$ be the total infiltration rates entering layers (1) and (2). The continuity equation is

$$dQ_x/dx = N \tag{85}$$

and it follows with (83) that

$$\frac{dQ_x^{(1)}}{dx} = \lambda^{(1)}N = N^{(1)} \qquad \frac{dQ_x^{(2)}}{dx} = \lambda^{(2)}N = N^{(2)} \tag{86}$$

The inflow into layer (1) through the upper boundary is $N^{(1t)} = N$, so that the inflow into layer (1) through the interface with layer (2) is $N^{(1b)}$, with

$$N^{(1b)} = N^{(1)} - N^{(1t)} = \lambda^{(1)}N - N = -(1-\lambda^{(1)})N \tag{87}$$

The expression for the streamline in layer (1) through (x_0, $H^{(1)} + H^{(2)}$) is obtained by the use of (29), where $Z(s)$ is to be replaced by $y - H^{(2)}$ (note that Z is measured from the bottom of the layer), $h(s)$ by $H^{(1)}$, N by $N^{(1)}$, $N^{(b)}$ by $N^{(1b)}$, Q_s by $Q_x^{(1)}$, and s by x,

$$\frac{y - H^{(2)}}{H^{(1)}} = e^{-f(x)}\left[1 - \int_{x_0}^{x} \frac{(1-\lambda^{(1)})N}{\lambda^{(1)}Nx} e^{f(x)}\, dx\right] \tag{88}$$

where (compare (27))

$$f(x) = \int_{x_0}^{x} \frac{N^{(1)}}{N^{(1)}x}\, dx = \ln \frac{x}{x_0} \tag{89}$$

Substitution of (89) for $f(x)$ in (88) yields, using (84),

$$\frac{y - H^{(2)}}{H^{(1)}} = -\frac{k^{(2)}H^{(2)}}{k^{(1)}H^{(1)}} + \frac{T}{k^{(1)}H^{(1)}}\frac{x_0}{x} \tag{90}$$
$$H^{(2)} \leqq y \leqq H^{(1)} + H^{(2)}$$

which is identical to (79). An expression for the streamline in layer (2) is obtained in a similar fashion. Let $x_0^{(2)}$ be the x coordinate of the point of intersection of the streamline (90) with the interface between the layers. Then $x_0^{(2)}$ is found by setting y equal to $H^{(2)}$ in (90), which gives

$$x_0^{(2)} = \frac{T}{k^{(2)}H^{(2)}} x_0 \tag{91}$$

Infiltrated water enters the lower layer only through the interface:

$$N^{(2)} = N^{(2t)} = \lambda^{(2)}N \qquad N^{(2b)} = 0 \tag{92}$$

Application of (29), with (27), yields, replacing $Z(s)$ by y, $h(s)$ by $H^{(2)}$, s_0 by $x_0^{(2)}$, N by $\lambda^{(2)}N$, $N^{(b)}$ by 0, and Q_s by $Q_x^{(2)}$ and using (83), (84), (91), and (92)

$$\frac{y}{H^{(2)}} = \frac{x_0^{(2)}}{x} = \frac{T}{k^{(2)}H^{(2)}}\frac{x_0}{x} \qquad 0 \leqq y \leqq H^{(2)} \tag{93}$$

The equation is, again, identical to that obtained from the exact solution, (80).

The good results are obtained for the two examples given

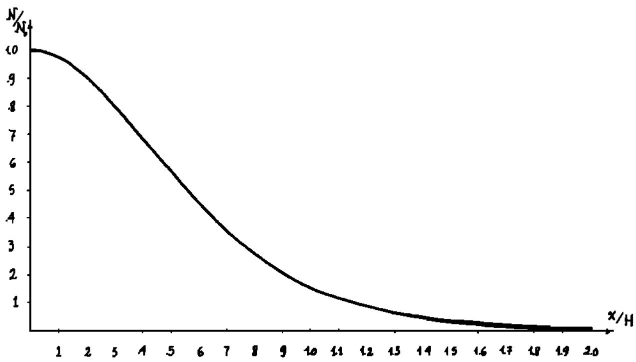

Fig. 11. Plot of N/N_0 versus x/H.

above because the rate of infiltration is constant for each case. The approximated streamlines will deviate from the exact ones when the rate of infiltration varies with position, as is shown below.

Exact Solution for Two-Dimensional Flow With a Variable Rate of Infiltration

The problem discussed below is similar to that of Figure 1, except that the rate of infiltration, $N = N(x)$, varies with position as follows:

$$N = \frac{N_0}{\{\cosh [\pi x/(2H)]\}^2} \tag{94}$$

Thus N has its maximum value at $x = 0$, $N(0) = N_0$ and decreases to zero at $x = \infty$; an inflection point occurs at $x = 0.419H$. A plot of N/N_0 versus x/H is shown in Figure 11. The x, y coordinate system is chosen with x pointing along the impervious base and y vertically upward along the left-hand impermeable boundary. The boundary conditions are

$$\begin{aligned} 0 \leqq x < \infty \quad & y = 0 \quad q_y = 0 \\ x = 0 \quad & 0 \leqq y \leqq H \quad q_x = 0 \\ 0 \leqq x < \infty \quad & y = H \quad q_y = -N(x) \end{aligned} \tag{95}$$

The solution to this problem may be written in terms of the following complex potential:

$$\Omega = -\frac{2}{\pi} N_0 z \coth \frac{\pi z}{2H} + C \tag{96}$$

where C is a real constant. The specific discharge function $w = -d\Omega/dz$ is obtained from (96) as

$$w = q_x - iq_y = \frac{2N_0}{\pi}\left[\coth \frac{\pi z}{2H} - \frac{\pi z}{2H}\left(\sinh \frac{\pi z}{2H}\right)^{-2}\right]$$

$$= \frac{N_0}{\pi}\left[\sinh \frac{\pi z}{H} - \frac{\pi z}{H}\right]\left[\sinh \frac{\pi z}{2H}\right]^{-2} \tag{97}$$

The reader may verify that the three boundary conditions (95) are met, noticing that the singularity of (97) at $z = 0$ is removable; w is zero there. Equations for the streamlines are obtained by setting the imaginary part of (96) equal to a constant. The starting point x_j of a streamline with $\Psi = \Psi_j$ is found as follows. The expression for Ψ for $z = x_j + iH$ is obtained from (96) as

$$z = x_j + iH \qquad \Psi_j = -\frac{2}{\pi} HN_0 \tanh \frac{\pi x_j}{2H} \tag{98}$$

so that

$$\frac{x_j}{H} = \frac{2}{\pi} \operatorname{arctanh}\left[-\frac{\pi \Psi_j}{2N_0 H}\right] \tag{99}$$

As an illustration the flow net obtained by the use of (96) is shown in Figure 12. Points of the streamlines are determined numerically by the use of a Newton-Raphson procedure, as discussed in the appendix. Travel times are computed in terms of the dimensionless time t^*, defined as

$$t^* = tN_0/nH \tag{100}$$

where n is the porosity. The time necessary for a fluid particle to travel from a point $s = s_0$ to s along a streamline defined by $x = x(s)$, $y = y(s)$ may be expressed as

$$t^* - t_0^* = \int_{s_0}^{s} \frac{N_0}{H} \frac{ds}{q_s} \tag{101}$$

where s is the arc length and q_s is the specific discharge along the streamline. Equation (101) is evaluated numerically, and points of equal travel times are marked on the streamlines drawn as solid curves in Figure 13. The dashed-dotted curves are isochrones and are obtained by connecting the marks.

Approximate Solution for Flow With a Variable Rate of Infiltration

The differential equation for the discharge Q_x for the flow problem of Figure 12 is

$$dQ_x/dx = N(x) \tag{102}$$

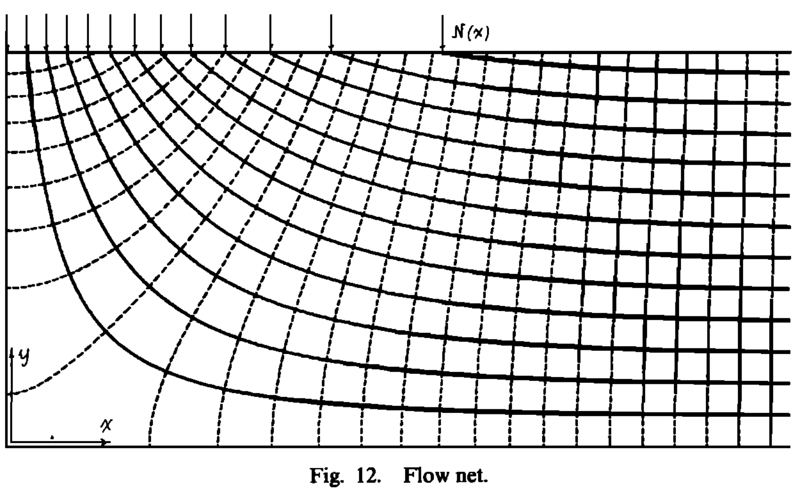

Fig. 12. Flow net.

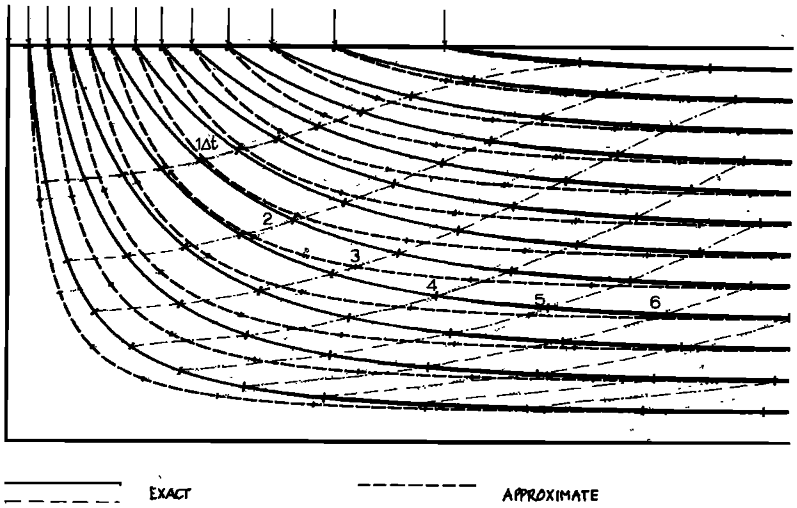

Fig. 13. Comparison of exact and approximate streamlines for $0 \leq x/H \leq 2$.

with $N(x)$ given by (94). Integration yields

$$Q_x = \frac{2N_0H}{\pi} \tanh \frac{\pi x}{2H} \tag{103}$$

The expression for the streamline through $x = x_j$, $y = H$ is obtained from (29) by the use of (102) and (103), which yields

$$\frac{y}{H} = \frac{\tanh [\pi x_j/(2H)]}{\tanh [\pi x/(2H)]} \tag{104}$$

The streamlines obtained from the approximate solution are shown in Figure 13 as the dashed curves. Travel times may be calculated by noticing that the velocity component v_x depends only upon x. Therefore travel time for any streamline can be calculated from

$$t^* - t_0^* = \int_{x_0}^{x} \frac{N_0}{H} \frac{dx}{q_x} \tag{105}$$

With $Hq_x = Q_x$ and using (103), the integration may be carried out:

$$t^* - t_0^* = \ln \frac{\sinh [\pi x/(2H)]}{\sinh [\pi x_0/(2H)]} \tag{106}$$

Points of equal travel times are marked on the approximate streamlines with marks that are half the size of those on the exact streamlines. The isochrones are drawn as dotted curves.

It is seen from Figure 13 that the approximate streamlines are consistently below the exact ones. A somewhat larger portion of the aquifer is shown in Figure 14. As with any approximate method, the magnitude of the errors will depend upon the problem; there will certainly be problems where the errors are unacceptable. For such cases, part of the flow domain may be modeled three dimensionally, whereby a transition to a Dupuit-Forchheimer model can be made where the three-dimensional effects become less pronounced.

Conclusion

The approximate elevation of streamlines may be computed for Dupuit-Forchheimer models as explained in this paper. The approach is based upon the interpretation of the Dupuit-Forchheimer assumption that the resistance to flow in the vertical direction is neglected. Its accuracy appears to depend primarily upon the variation with position of the rate of infiltration through the upper and lower boundaries of the aquifer considered. The approach may be used in computer models based on either numerical or analytic techniques. It is applicable to aquifers with layers of different permeabilities and will be particularly useful in determining flow paths in systems of stacked aquifers. It is expected that in many problems of practical importance the approximate method of determining streamlines in three dimensions will make it unnecessary to resort to three-dimensional models.

Appendix: Tracing Streamlines Using the Stream Function

Let a point $z = z_0$ lie on the streamline $\Psi = \Psi_0$. The first estimate $z_1^{(1)}$ of the next point on the streamline may be made as follows (see Figure A1):

$$z_1^{(1)} - z_0 = \frac{\overline{w(z_0)}}{n} \Delta t \tag{A1}$$

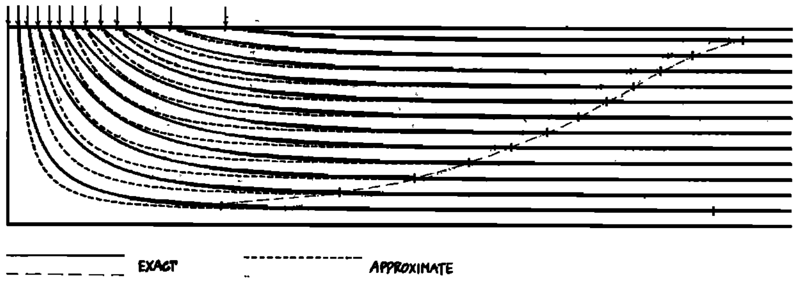

Fig. 14. Comparison of exact and approximate streamlines for $0 \leq x/H \leq 4$.

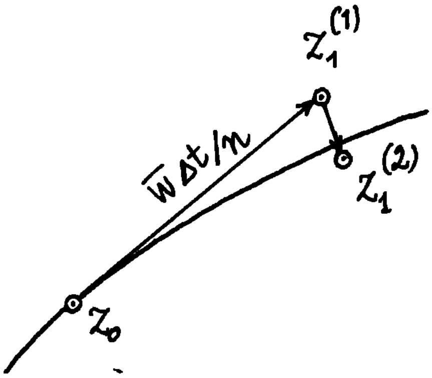

Fig. A1. Calculation of points on a streamline.

where Δt is a fixed increment, n is the porosity, and the bar denotes the complex conjugate. The next approximation of z_1 is found by the application of the Newton-Raphson procedure along the equipotential $\Phi = $ const through $z_1^{(1)}$. With

$$w = -\frac{d\Omega}{dz} \sim -\frac{\Delta\Phi + i\Delta\Psi}{z_1^{(2)} - z_1^{(1)}} = -i\,\frac{\Psi_0 - \Psi(z_1^{(1)})}{z_1^{(2)} - z_1^{(1)}} \qquad \text{(A2)}$$

The following recursive relation is obtained for a new approximation $z_1^{(j+1)}$ in terms of the old one $z_1^{(j)}$:

$$z_1^{(j+1)} = z_1^{(j)} + i\,\frac{\Psi(z_1^{(1)}) - \Psi_0}{w(z_1^{(j)})} \qquad \text{(A3)}$$

where it is noted that stagnation points ($w = 0$) are to be avoided. The time increment Δt in (A1) is a first approximation of the time of flow from z_0 to z_1. Better approximations may be made by numerical integration after z_1 is found.

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